Set theoretical pathologies in the problem of Lyapunov stability of singular points of vector fields

Yu. Ilyashenko

Abstract. We prove that Lyapunov stability problem demonstrates pathologies even on the set-theoretical level. Namely, there exists an analytic one-parameter family of 5-jets of vector fields that crosses the set of stable jets by a countable union of disjoint intervals.

Key words Lyapunov stability, stable jets, factor system, radial function

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1. Introduction

In 1970 Arnold (and independently R.Thom) introduced a notion of algebraically solvable local problem of analysis. In [A] Arnold proved that the problem of Lyapunov stability of singular points of vector fields is algebraically unsolvable. He conjectured: "One may expect that the Lyapunov stability problem having lost the algebricity and no more restricted by anything will demonstrate pathologies even on the set-theoretical level. For instance, the set of the stable jets in a finite-dimensional algebraic subset of the space of jets of a fixed order may plausibly have an infinite number of connected components or be dense together with its complement." For this Arnold suggested to make use of the resonances on the two-torus. Following this way, the author succeeded to prove analytic non-solvability of the Lyapunov stability problem [I]. In the present paper, first part of the Arnold's conjecture is proved, yet not for the algebraic, but rather for analytic families, see Theorem 1 below.

This paper has much in common with [I] (including the quotation above), but many differences and simplifications as well. So we do not refer to technical details of [I], but rather give the independent presentation.

In order to state our result, let us recall some definitions.

All the vector fields below are C^∞. An N-jet of a vector field v in $(\mathbb{R}^n, 0)$ is a set of all vector fields that differ from v by $o(|x|^N)$ at 0. Notation $v^{(N)}$. Vector fields from this set are called representatives of the jet. The set of all N-jets of vector fields in $(\mathbb{R}^n, 0)$ is denoted by $J^{N,n}$. It is a finite dimensional linear space.

A jet of a vector field for which all the representatives are Lyapunov stable (unstable) is called Lyapunov stable (unstable) jet. A jet of a vector field that has both stable and unstable representatives is called neutral.

The main result of this paper is the following.

Theorem 1. *There exists a one-parameter analytic family of jets in $J^{5,5}$ that intersects the set of Lyapunov stable jets by a disjoint union of an infinite number of intervals.*

2. A special class of jets of homogeneous vector fields

2.1. Factor systems and radial functions

Begin with some known definitions. Let v be a homogeneous vector field in $\mathbb{R}^{n+1}$, and

$$\dot{u} = v(u) \tag{1}$$

be the corresponding equation. We may replace the singular point by the projective space with a vector field on its affine neighborhood that determines a so called factor system. This is sort of desingularizstion. The factor system may be arbitrary complicated. This shows a connection between the local and global theories of differential equations. This connection was successfully used by Arnold, Takens and others in the 70s and later on.

Recall the definition of factor systems and radial functions.

For any $a = (a_0, ..., a_n)$ let $a = (a_0, \tilde{a})$. For $u \in \mathbb{R}^{n+1}$, let $x = \frac{\tilde{u}}{u_0}$; x is a chart on an affine neighborhood $u_0 \neq 0$ of $\mathbb{R}P^n$. Set $r^2 = (u, u)$. Let v be a homogeneous polynomial vector field in $\mathbb{R}^{n+1}$. For a suitable time change, the homogeneous equation $\dot{u} = v(u)$ implies a factor system in $\mathbb{R}P^n$: $x' = V(x)$ and a radial equation $r' = R(x)r$; R is called a radial function.

Lemma 1. *For any polynomial P of degree $N+1$ and any polynomial vector field V of degree N , both in $\mathbb{R}^n$, there exists a homogeneous vector field v of degree $N+2$ in $\mathbb{R}^{n+1}$ such that V defines a factor-system for v, and the corresponding radial equation, after an appropriate time change, has the form*

$$r' = R(x)r \tag{2}$$

$$R = \frac{P}{1 + (x, x)}. \tag{3}$$

Proof. We will construct v in the form $v = v^1 + v^2$, v^1 having the desired factor system, and zero radial function, v^2 having the desired radial function, and zero factor system. Take

$$v^1(u) = f(u)(u, u) - (f, u)u,\ f = (0, \tilde{f}),\ \tilde{f} = u_0^N V\left(\frac{\tilde{u}}{u_0}\right).$$

Obviously,

$$(v^1(u), u) \equiv 0,\ \dot{x} = \left(\frac{\tilde{u}}{u_0}\right)^{\cdot} = u_0^{N-1} V\left(\frac{\tilde{u}}{u_0}\right)(u, u).$$

Take

$$v^2(u) = u_0^{N+1} P\left(\frac{\tilde{u}}{u_0}\right)u.$$

Obviously, for this vector field

$$\dot{x} \equiv 0,\ \dot{r} = \frac{u_0^{N-1} P\left(\frac{\tilde{u}}{u_0}\right)(u, u)}{1 + (x, x).}$$

The time change

$$\frac{d}{dt} = {}^{\cdot} = u_0^{N-1}(u, u)\frac{d}{d\tau},\ \frac{d}{d\tau} = {}'$$

implies

$$x' = V(x),\ r' = \frac{P(x)}{1 + (x, x)}r.$$

□

2.2. Stable jets of special homogeneous vector fields

In this subsection dimension of the phase space is arbitrary.

Theorem 2. *Let v be a polynomial homogeneous vector field of degree N. Let*

$$x' = V(x) \tag{4}$$

be the corresponding factor system. Suppose that

(1) The factor system has an attracting compact invariant manifold M and an absorbing neighborhood $U \supset M$ with a compact closure and with a Lyapunov function W :

$$W|_{U\setminus M} > 0,\ W|_M = 0,\ W \in C^1, L_V W|_{U\setminus M} < 0.$$

(2) Let the radial function R be negative outside U and tend to $-\infty$ at infinity.

(3) Let g_V^τ be the time τ phase flow transformation of the factor system (4), and

$$R_T(x) = \frac{1}{T}\int_0^T R\circ g^\tau(x)d\tau,\ \bar{R}(x) = \overline{\lim}_{T\to\infty} R_T(x).$$

Then

if $\bar{R} < \delta < 0$ on M, then the jet $v^{(N)}$ is stable;

if $R(x) > 0$ for some $x \in M$, then the jet $v^{(N)}$ is unstable or neutral.

Begin with a lemma.

Lemma 2. *Let in assumption of the theorem, $\bar{R} < \delta < 0$ on M. Then there exists $T > 0$ such that for any $y \in U$*

$$R_T(y) < \frac{\delta}{2}. \tag{5}$$

2.3. Proof of Lemma 2

By assumption of the lemma, for any $x \in M$ there exists T_x such that

$$\bar{R}_{T_x}(x) < \frac{2}{3}\delta.$$

By the continuous dependense theorem, there exists a neighborhood $U_x \subset U$ of x such that $R_{T_x}(y) < \frac{2}{3}\delta$ for any $y \in U_x$. The family $\{U_x | x \in M\}$ forms a cover of M. Let us take a finite subcover of this cover by the neighborhoods U_{x_j} such that $R_{T_{x_j}}(y) < \frac{2}{3}\delta$ for any $y \in U_{x_j}$. Let us redenote for brevity $U_{x_j} = U_j, T_{x_j} = T_j$. Let $U_0 = \cup U_j$. This is a neighborhood of

M. As M is attracting in U, all the points of U enter U_0 after some positive time. Let T^* be such that

$$g_V^{T^*}(U) \subset U_0.$$

Let $m = \max_{\bar{U}} R^+$, $R^+ = \max(R, 0)$, $T_0 = \max T_j$. Take T such that

$$T > \left(4 - \frac{6m}{\delta}\right)(T_0 + T^*). \tag{6}$$

Let us prove that for this T inequality (5) holds. Take any $y \in U$, and split the time segment $[0, T]$ by the points t_j constructed inductively. Let $t_1 \leq T^*$ be such that

$$g_V^{t_1}(y) := x_{(1)} \in U_0.$$

The point $x_{(1)}$ belongs to some neighborhood $U_{(1)}$. Take the corresponding time $t_2' = T_{x_{(1)}}$ and let

$$x_{(2)} = g^{t_2'}(x_{(1)}),\ t_2 = t_1 + t_2'.$$

In the same way, the time moments t_j and the points $x_{(j)}$ are constructed. Let t_k be such that $T - t_k < T_0$. Then

$$\int_0^T R \circ g_V^T(y)dt = \int_0^{t_1} R \circ g_V^T(y)dt + \sum_2^k \int_{t_{m-1}}^{t_m} R \circ g_V^t(y)dt$$

$$+ \int_{T_k}^T R \circ g_V^t(y)dt := I_1 + I_2 + I_3.$$

We have:

$$I_1 + I_3 < m(T^* + T_0),\ I_2 < \frac{2}{3}\delta(t_k - t_1).$$

Hence,

$$T\bar{R}_T(y) < \frac{2}{3}\delta T + (m - \frac{2}{3}\delta)(T^* + T_0).$$

For T from (6),

$$T\bar{R}_T(y) < \frac{1}{2}\delta T$$

This proves the lemma.

2.4. Proof of Theorem 2

Let $\rho = \ln r$. In the coordinates (x, ρ) the system (4), (2) takes the form:

$$\begin{pmatrix} x' \\ \rho' \end{pmatrix} = \begin{pmatrix} V(x) \\ R(x) \end{pmatrix} := w(x). \tag{7}$$

A perturbation of the system (1) with the same N-jet at zero has the form

$$\dot{u} = v(u) + \hat{v}(u),\ \hat{v}(u) = O(|u|^{N+1}).$$

In the (x, ρ) coordinates, after appropriate time rescaling, this system takes the form:

$$\begin{pmatrix} x' \\ \rho' \end{pmatrix} = w(x) + \hat{w}(x, \rho), \tag{8}$$

where

$$|\hat{w}| < C \exp(-\rho), \rho < \beta$$

for some $C > 0, \beta \in \mathbb{R}$.

Let U, V and M be the same as in Theorem 2, and T be the same as in Lemma 2. Let $U_1 = g_V^T U$. Then $U_1 \subset U$ because U is absorbing. Consider a cylinder

$$Z_0 = U_1 \times \{\rho < \beta\}.$$

It is tempting to claim that for an appropriate β this cylinder is absorbing for the system (8). But this is wrong in general. Instead, we extend the system (8) to Z_0 multiplied by a factorized time axis $S^1 = \mathbb{R}(modT)$ and find a time depending Lyapunov function there. Begin with the unperturbed system. Let

$$Z_1 = U_1 \times \mathbb{R} \times S^1.$$

For any vector field v on Z_0 denote by $\tilde{v}$ its extension to Z_1 obtained by adding the time-component 1:

$$\begin{pmatrix} v(x) \\ 1 \end{pmatrix} = \tilde{v}(x, \tau)$$

Consider the unperturbed system (7) lifted to Z_1:

$$\begin{pmatrix} x' \\ \rho' \\ t' \end{pmatrix} = \tilde{w}.$$

Let us construct a first integral F of this system defined in Z_1. For any $(x, \rho, \tau) \in Z_1$ take a point

$$(x(0), \rho(0), 0) = g_{\tilde{w}}^{-\tau}(x, \rho, \tau).$$

Let

$$F(x,\rho,\tau)=\rho-\int_0^\tau R\circ g_{\tilde w}^{t-\tau}(x,\rho,\tau)dt.$$

We have

$$L_{\tilde w}F=0.$$

This integral has a jump on the section $\tau=0(modT)$:

$$J(x)=\int_0^T R\circ g_{\tilde w}^{t-T}(x,\rho,T)dt.$$

Note that R does not depend on ρ, and $g_{\tilde w}^t$ commutes with the shift along the ρ-axis; hence, J depends indeed on x only. By definition of F,

$$F(x,\rho,T)-F(x,\rho,0)=-J(x).$$

Let us take

$$W_0(x,\rho,\tau)=F(x,\rho,\tau)+\frac{\tau}{T}J(x).$$

This is a piecewise C^1 continuous function on Z_1, and

$$L_{\tilde w}W_0=\frac{J}{T}<\frac{\delta}{2}<0;$$

the first inequality follows from Lemma 2. The function W_0 is a Lyapunov function in the cylinder Z_1 for the unperturbed system (7). Hence, the set $Z_1\cap\{W_0<C\}$ is absorbing for the system (7) for any $C\in\mathbb{R}$.

Now let us turn to the perturbed system (8). For $\beta<0, |\beta|$ sufficiently large, the term $\hat w$ in (8) is small in Z_1. Hence, in $Z_1\cap\{W_0<-C\}$ for C sufficiently large,

$$L_{\widetilde{w+\hat w}}W_0<0,$$

and the cylinder $Z_1\cap\{W_0<-C\}$ is absorbing for the system (8). Hence, though the cylinder Z_0 is not absorbing for the perturbed system (8), but for β large the function ρ along the orbits that start in Z_0 uniformly tends to $-\infty$ as $t\to\infty$.

Now let us prove the same for the cylinder

$$Z_2=U\times\{\rho<\alpha\}\times S^1$$

for $\alpha<0$, where $|\alpha|$ depending on β is large enough.

Let W be the Lyapunov function of the factor system (4) in U. Let $W^*(x,\rho,\tau)=W(x)$. The derivative $L_{\widetilde{w+\hat w}}W^*$ is strictly negative in $Z_2\setminus Z_0$ for α large, hence, orbits of system (8) reach

Z_0 from Z_2, and the function ρ along the orbits that start in Z_2 uniformely tends to $-\infty$ as $\tau \to \infty$.

Consider at last the domain $(\mathbb{R}^n \setminus U) \times (\rho < \beta)$. In this domain

$$\rho' = R + \hat{R}, R < -\gamma$$

for some $\gamma > 0$ by assumption (2) of the theorem; $\hat{R} < ce^{\rho}$ for $\rho < -\beta, \beta$ is large.

This proves that $v^{(N)} \in J^+$. Theorem 2 is proved.

3. Special one parameter family that crosses the set of stable jets by a countable union of disjoint intervals

3.1. Auxiliary two-parameter family

We will construct a family $\mathcal{V}$ of homogeneous polynomial vector fields $v_{a,\omega}$ of degree 5 in $\mathbb{R}^5$ depending on two real parameters a, ω and satisfying the following assumptions.

1. All the vector fields $v_{a,\omega}$ satisfy the assumptions of Theorem 2. Namely, the attracting manifold of all the factor-systems will be one and the same two-torus with the natural angle coordinates φ, ψ that satisfy the equation

$$\varphi' = Q, \ \psi' = \omega Q, \tag{9}$$

where Q is a trigonometric polynomial positive everywhere on $\mathbb{T}^2$.

2. The radial function R on $\mathbb{T}^2$ has the form

$$R_{a,\omega}(\varphi, \psi) = a + P(\varphi, \psi),$$

where P is a trigonometric polynomial, and

$$\Phi_{a,\omega}(\varphi, \psi) := \frac{P}{Q}(\varphi, \psi) = \sum_{m,n=0}^{\infty} a_{m,n} \sin(m\varphi - n\psi),$$

the series absolutely converges.

Moreover, for any prime $\frac{p}{q} \in \mathbb{Q}$

$$\Psi_{p,q} := \sum_{k=0}^{\infty} a_{kp,kq} \sin k(p\varphi - q\psi) \not\equiv 0, \tag{10}$$

that is, at least one coefficient in the series above is non-zero.

Theorem 3. *There exists a one-parameter analytic subfamily $A_1 \subset A$ that crosses the set of stable jets by a countable union of disjoint intervals.*

This theorem immediately implies Theorem 1. It is proved below.

3.2. Stability boundary in the special two-parameter family

We consider the 5-jets of the vector fields $v_{a,\omega}$. Stable and neutral or unstable jets from this family are described by Theorem 2. The phase curves of the system (9) are the windings of the torus, given by the formula

$$\psi = \omega\varphi + \psi_0.$$

They are dense for irrational ω and closed for rational ones. The T time average of R over these curves is given by

$$R_T(\varphi(0), \psi(0)) = \frac{1}{T}\int_0^T (a + P(\varphi(t), \omega\varphi(t) + \psi_0)dt =$$

$$a + \frac{1}{T}\int_{\varphi(0)}^{\varphi(T)} \frac{P}{Q}(\varphi, \omega\varphi + \psi_0)d\varphi =$$

$$a + \frac{1}{T}\int_{\varphi(0)}^{\varphi(T)} \Sigma a_{mn} \sin(m\varphi - n\omega\varphi - n\psi_0)d\varphi$$

For ω irrational, the average of the integral tends to zero, and $\bar{R} := \overline{\lim}_{T\to\infty} R_T = a$. For $\omega = \frac{p}{q}$ prime, the terms with $m - \frac{p}{q}n \neq 0$ tend to zero:

$$\lim_{T\to\infty} \frac{1}{T}\int_{\varphi(0)}^{\varphi(T)} \sin((m - \frac{p}{q}n)\varphi - n\psi_0)d\varphi = 0.$$

Note that $\psi - \frac{p}{q}\varphi$ is constant along the orbit, and this constant equals ψ_0. Hence, for $\omega = \frac{p}{q}, \psi_0 = \psi - \frac{p}{q}\varphi$,

$$R_{a,\omega}(\varphi, \psi) = a + \lim \frac{\varphi(T)}{T} \sum a_{pk,qk} \sin qk(\psi - \frac{p}{q}\varphi) := a + r_{p,q}(\psi_0).$$

It is easy to prove that this limit exists. The function $r_{p,q}$ is odd. Let $\mu(\frac{p}{q})$ be its (negative) minimum. Obviously, 0 is the only accumulation point of the set $\{\mu(\frac{p}{q}) | \frac{p}{q} \in \mathbb{Q}\}$.

Denote by $\sigma(\frac{p}{q})$ the segment $[0, \mu(\frac{p}{q})]$. The function

$$a(\omega) = \begin{cases} 0 \text{ for } \omega \notin \mathbb{Q} \\ \mu(\frac{p}{q}) \text{ for } \omega = \frac{p}{q} \end{cases}$$

resembles the Riemann function. The union

$$\Gamma = \{a = 0\} \cup (\cup_{\frac{p}{q} \in \mathbb{Q}} \sigma_{\frac{p}{q}}) \tag{11}$$

is the subgraph of $a(\omega)$. (It is a subgraph according to the general definition, though Γ lies above the graph of $a(\omega)$).

Now let us apply Theorem 2. For $a > 0$, the function $R_{a,\omega}$ admits positive values on $\mathbb{T}^2$, hence the jet $v^{(N)}$ is unstable or neutral.

For $a < 0$, the function $R_{a,\omega}$ equals a for ω irrational, and takes values from $a + \sigma(\frac{p}{q})$ for $\omega = \frac{p}{q}$. Thus the set of stable jets in the family $\mathcal{V}$ has the form:

$$\{(a,\omega) | v^{(N)}_{a,\omega} \in J^+\} = \{a < 0\} \setminus \Gamma.$$

Therefore, this intersection is not by far a semi-analytic set. This implies analytic non-solvability of the Lyapunov stability problem proved first in [I].

Let us turn to the last step in the proof of Theorem 3.

3.3. Special one-parameter subfamily

The desired subfamily will be constructed in the form

$$A_1 = \{(a,\omega) | a = \omega - \omega_0\}$$

in such a way that ω_0 will be approximated by a countable number of rationals with the accuracy better than $|\mu(\frac{p}{q})|$. For such rationals, the line A_1 crosses the segment $a \times \sigma(\frac{p}{q}) \subset \Gamma$ that belongs to the complement of J^+. The construction of ω_0 is similar to the Liouvelle construction of the transcendental numbers.

Let us define by induction a seqence of nested segments s_k. Take any $r_1 = \frac{p_1}{q_1} \in (0,1)$ and let

$$s_1 = [r_1 + \mu(r_1), r_1].$$

By induction, take any $r_k = \frac{p_k}{q_k} \in s_{k-1}$ and let

$$s_k = s_{k-1} \cap \{[r_k + \mu(r_k), r_k]\}.$$

The sequence of segments s_k is nested and shrinking. Take ω_0 as its intersection point. The line A_1 intersects all the segments $\{r_k\} \times \sigma(r_k)$ for r_k defined above. The family A_1 is constructed.

3.4. An explicit construction of the family $\mathcal{V}$

To construct a family $\mathcal{V}$ of homogeneous vector fields of degree 5 in $\mathbb{R}^5$ with the properties listed above, we apply Lemma 1 and just start with a polynomial factor-system of degree 3 and a radial function of degree 4. The phase space of the factor system is $\mathbb{R}^4$ with the coordinates

$$z_1 = x_1 + ix_2 = \rho_1 e^{i\varphi},\ z_2 = x_3 + ix_4 = \rho_2 e^{i\psi}.$$

Let for brevity $\frac{1}{3} = \lambda$. Let

$$F = \lambda(x_2 + x_4) = Im\lambda(z_1 - \bar{z}_2),$$

$$G = |1 - \lambda(z_1 - \bar{z}_2)|^2$$

Let $h_j = 1 - \rho_j^2$,

$$V_{a,\omega} = (iz_1 G + z_1 h_1,\ iz_2\omega G + z_2 h_2)$$

$$R_{a,\omega} = a + F - C(h_1^2 + h_2^2)$$

The factor system has the form

$$z_1' = iz_1 G + z_1 h_1, z_1' = iz_2\omega G + z_2 h_2,$$

Let us prove that for C large enough, the factor system and the radial function satisfy the assumptions of Section 3.1.

For any function $H(z_1, z_2)$ let H^* be the same function written in the polar coordinates:

$$H^*(\rho_1, \varphi_1, \rho_2, \varphi_2) = H(\rho_1 e^{i\varphi_1}, \rho_2 e^{i\varphi_2}).$$

In the polar coordinates, the factor system has the form

$$\begin{cases} \rho_1' = \rho_1(1 - \rho_1^2), \rho_2' = \rho_2(1 - \rho_2^2) \\ \varphi' = G^*, \psi' = \omega G^*. \end{cases} \tag{12}$$

This implies assumption 1 of Section 3.1.

Let $W = (1 - \rho_1)^2 + (1 - \rho_2)^2$, $U = \{W < \frac{1}{2}\}$. The domain U is absorbing for the system (12) with the Lyapunov function W.

We have:

$$R|_{\mathbb{T}^2} = a + P,\ \frac{P}{Q} = Im\frac{1}{1 - \lambda(e^{i\varphi} - e^{-i\psi})}$$

Let us check assumption 2 of Section 3.1.

$$\frac{P}{Q} = Im\sum \lambda^k(e^{i\varphi} - e^{-i\psi})^k = \sum \lambda^{m+n}(-1)^n C_{m+n}^n \sin(m\varphi - n\psi).$$

As $\lambda = \frac{1}{3}$, this series is absolutely convefging, and has non-zero coefficients. This implies assumption 2 of Section 3.1.

The desired example is constructed. Theorem 3, hence 1 is proved.

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Yu. Ilyashenko, National Research University Higher School of Economics, Russian Federation; Independent University of Moscow

yulijs@gmail.com

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Yu. Ilyashenko