DULAC MAPS OF REAL SADDLE-NODES

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ABSTRACT. Consider a germ of a holomorphic vector field at the origin on the coordinate complex plane. This germ is called a saddle-node if the origin is its singular point, one of its eigenvalues at zero is zero, and the other is not. A saddle-node germ is real if its restriction to the real plane is real. The monodromy transformation for this germ has a multiplier at zero equal to 1. The germ of this map is parabolic and admits a "normalizing cochain". In this note we express the Dulac map of any real saddle-node up to a left composition with a real germ $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ through one component of the cochain normalizing the monodromy transformation.

Key words saddle-node, monodromy, Ecalle-Voronin and Martinet-Ramis moduli

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1. Introduction

In 1981 Voronin [V], Ecalle [E], and Malgrange [M] discovered functional invariants of the analytic classification of parabolic germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, the germs with the linear part the identity. These invariants are now called Ecalle-Voronin moduli (Malgrange in the title of his talk [M] at the Bourbaki seminar referred to Ecalle, though his approach was different). In words of Arnold (oral communication) this discovery revolutionized the theory of normal forms. The effect discovered is called "Nonlinear Stokes Phenomena", see for instance the book [I2]. In the last decades this phenomenon was investigated by Christopher, Glutsyuk, Loray, Mardesic, Roussarie, Rousseau, Shishikura and others, see [G], [LTV], [MRR], [RC], [S] and many more.

Martinet and Ramis [MR] gave an analytic classification of complex saddle nodes (germs of vector fields in $(\mathbb{C}^2, 0)$ with one zero and one non-zero eigenvalue). The Martinet–Ramis moduli of this classification are at the same time Ecalle-Voronin moduli of the monodromy map of the saddle-node. (The definition of the monodromy map as well as the other special terms of the Introduction are recalled in the next section.)

This paper is devoted to the study of the Dulac map of a real saddle-node. The formula for this map, based on [MR], may be found in [I1], see formula (10) below. This formula contains three factors. For the non-real saddle-nodes the third factor (counted from the right) may be eliminated by a non-real coordinate change in the target. For the real saddle-nodes only real coordinate changes are admitted. The third factor remains non-trivial, though not uniquely determined. Seemingly, the first and the third factors are quite independent. The present paper shows that for real saddle-nodes there is a quite unexpected relation between the two factors, and the whole product may be in a sense reconstructed from the last factor.

Real saddle-nodes are the only elementary non-hyperbolic singular points of real vector fields. By the desingularization theorem, any isolated singular point of a planar analytic vector field may be split to a finite number of elementary ones by a finite number of blowing ups. So the elementary singular points play a role of physical elements in building of the boundless variety of the phase portraits of planar vector fields. Therefore every new property of saddle-nodes is important for the qualitative theory of differential equations as a whole. Moreover, Dulac maps of real saddle-nodes are crucial for the theory of the monodromy maps of real analytic polycycles, see [I1]. One may expect that the relations found in this paper may be applied to this theory, though right now these applications are not found.

2. Preliminaries and main results

2.1. The monodromy map and its formal normal form. Consider a saddle-node germ v of a vector field at a singular point 0 in $\mathbb{C}^2$ with the eigenvalues $\mu = 0 \neq \lambda$. As mentioned above, this germ is called a saddle-node. If $v|_{\mathbb{R}^2}$ is real then the saddle-node is real.

Choosing an appropriate coordinate and time change, one can write a saddle-node vector field in the form

$$\dot{x} = \pm x^{k+1} + \dots, \dot{y} = -y \dots, \tag{1}$$

dots replace the higher order terms. We consider those saddle-nodes for which the Dulac map is well defined, see Figure 1 and definitions below. For such vector fields the first equation in (1) has the form

$$\dot{x} = x^{k+1} + \dots$$

Any complex saddle-node has an invariant manifold, a holomorphic curve tangent at zero to the eigenvector of the linearization of the germ with the non-zero eigenvalue. A positive circuit of zero on this manifold

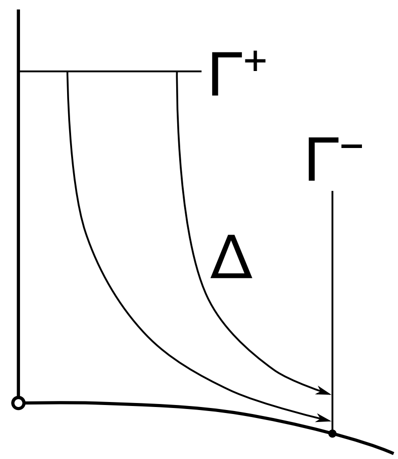

FIGURE 1. Dulac map of a real saddle-node

generates the monodromy map of this saddle-node denoted by m. This m is a parabolic germ: $m(0) = 0, m'(0) = 1$. Its formal normal form is

$$m_0 = g_{w_{k,a}}^{-2\pi i},\ w_{k,a} = \frac{z^{k+1}}{1 + az^k}\frac{\partial}{\partial z} \tag{2}$$

for some $k \in \mathbb{N}$, and $a \in \mathbb{C}$. For a real saddle-node a is real. Here g_w^t is the time t shift along the orbits of the vector field w (in other words, g_w^t is the time t phase flow transformation of the vector field w).

Important for the future applications is the rectifying chart of this field:

$$t_{k,a} = -\frac{1}{kz^k} + a \ln z. \tag{3}$$

Mention here that in [I1] a similar function is permanently used: $h_{k,a} = -1/t_{k,a}$.

2.2. **The Dulac map.** A real saddle-node v has a common separatrix of the two hyperbolic sectors that is not in general analytic at zero, see Fig 1. We can always choose the holomorphic invariant manifold as the w axis. Let Γ^+ be a cross-section $w = 1$ with the z coordinate on it, and $\Gamma^- = \{z = c\}$ be another cross-section. Changing the scale we can always achieve that the vector field v is well defined on these cross-sections. The map $\Delta : \Gamma^+ \to \Gamma^-$ along the orbits of v is called the Dulac map of the saddle-node v, see Figure 1.

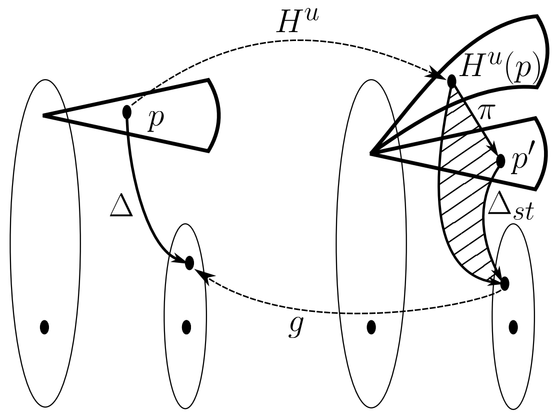

FIGURE 2. Sectorial normalization H^u and the Dulac map of a complex saddle-node

The following figure illustrates the Dulac map of a real saddle-node in the complex plane and formula (10) for this map. A detailed commentary to this figure is given in Section 2.6.

Our main goal is stated in the following theorem; the special terms used there will be explained below in this section.

Theorem 1. *The Dulac map for the real saddle-node may be expressed through the so called normalizing cochain for the monodromy map and the formal normal form of this map.*

The more explicit expression is presented below.

Recall some definitions.

2.3. **Normalizing cochains.** The formal series that conjugates a parabolic germ m with its formal normal form m_0, see (2), do diverge in general. The object that conjugates m and m_0 is a so called normalizing cochain that we now describe.

Definition 1. *A nice k-covering of a punctured disc $D^*_\varepsilon = \{z : 0 < |z| < \varepsilon\}$ is a covering of D^*_ε by sectors*

$$S_j = \{z : |z| < \delta, \left| \arg z - \frac{\pi}{2k} - \frac{\pi j}{k} \right| < \alpha. \ \alpha \in \left(\frac{\pi}{2k}, \frac{\pi}{k} \right) \};$$

the larger α is, the smaller is ε.

Definition 2. *A normalizing cochain $F = F_{norm}$ for a germ m with the formal normal form (2) is a tuple of holomorphic maps F_j such that :*

any F_j is defined in some sector S_j of the nice k-covering of the punctured disc;

all the F_j have the same asymptotic Taylor series at 0 that coincides with the formal series that formally conjugates m and m_0;

F conjugates m and m_0 analytically:

$$m_0 \circ F = F \circ m$$

whenever defined.

2.4. **Ecalle-Voronin and Martinet-Ramis moduli.** The sectors S_j, S_{j+1} and S_{2k}, S_1 overlap; in their pairwise intersection a compositional coboundary of $F = F_{norm}$ is defined:

$$\delta F = \{F_j \circ F_{j-1}^{-1}, j = 2, ..., 2k; F_1 \circ F_{2k}^{-1}\} \tag{4}$$

The compositional coboundary of the normalizing cochain, up to a simple equivalence relation that we skip, is the modulus of the analytic classification of the parabolic germs. It is called the Ecalle-Voronin modulus. The Ecalle-Voronin modulus of the monodromy transformation of a complex saddle-node is the modulus of the analytic classifiaction of complex saddle-nodes; it is called the Martinet-Ramis modulus.

The main objects for the description of the Dulac map for a saddle-node are the components F_1 and F_{2k} of the normalizing cochain and their compositional ratio. The sectors S_1 and S_{2k} both contain a germ $(\mathbb{R}^+, 0)$. The bisector of the sector $S_1 (S_{2k})$ is located above (below) the $\mathbb{R}^+$ semiaxis. For this reason we redenote:

$$F_1 = F^u_{norm},\ F_{2k} = F^l_{norm};$$

u and l stand for upper and lower. Sometimes we write F^u, F^l instead of $F^u_{norm},\ F^l_{norm}$.

At this spot we recall the main ingredient in the construction of the Ecalle–Voronin moduli. As mentioned above, these moduli are compositional coboundaries of the normalizing cochains. Namely, in the intersection of two adjacent sectors of the nice k-covering, we consider a compositional ratio of the components of the cochain corresponding to these sectors. Let us describe one component of the coboundary equal to $F^u \circ (F^l)^{-1}$. The components F^u, F^l conjugate the germ m to m_0. Let $\tilde{F}^u, \tilde{F}^l, \tilde{m}, \tilde{m}_0$ be the germs F^u, F^l, m, m_0 written in the rectifying chart $t_{a,k}$: $\tilde{F}^u = t_{a,k} \circ F^u \circ t_{a,k}^{-1}$, and so on. These are now germs at infinity. The germ $\tilde{m}_0$ is simply a shift $\zeta \to \zeta - 2\pi i$. The germs $\tilde{F}^u$ and $\tilde{F}^l$ both conjugate $\tilde{m}$ to $\tilde{m}_0$. Hence, their quotient, $\delta_0 = \tilde{F}^u \circ (\tilde{F}^l)^{-1}$ conjugates $\tilde{m}_0$ to itself, that is, commutes with $\tilde{m}_0$. Hence,

$$\delta_0 = \zeta + a_0 + \sum_1^\infty a_n \exp n\zeta$$

in some left halfplane

$$\mathbb{C}_b^- = \{\zeta : Re\zeta < b < 0\}.$$

The Martinet-Ramis modulus consists of the compositional coboundary of the normalizing cochain F for the monodromy map m. This coboundary belong to a special functional space $\mathcal{M}$ that we do not describe here. We mention only that any element of this space may be realized as a functional invariant for some parabolic germ of a map $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$. On the contrary, not any element from $\mathcal{M}$ may be realized as a Martinet-Ramis modulus, but rather that one that satisfies what may be called Martinet-Ramis restrictions. Equality $a_0 = 0$ in the formula for δ_0 is an example of these restrictions. In the description of the Dulac map we need only that component of the compositional coboundary of F that is defined in a sector that contains the germ $(\mathbb{R}^+, 0)$; this germ belongs to the crosse-section Γ^+.

So, according to the Martinet-Ramis theory, there is no free term in the series for δ_0, see the original paper [MR], or the survey [I2] (Corollary 2 in Section 3.4). Finally,

$$F^u \circ (F^l)^{-1} = (t_{k,a})^{-1} \circ \delta_0 \circ t_{k,a}, \tag{5}$$

$$\delta_0 = \zeta + \sum_1^\infty a_n \exp n\zeta. \tag{6}$$

Note that $\delta_0 - \zeta$ is $2\pi i$-periodic, and δ_0 commutes with the shift $\zeta \to \zeta - 2\pi i$. The germ δ_0 is a part (better to say, a component) of the Martinet-Ramis modulus. The difference $\delta_0 - \zeta$ is $2\pi i$ periodic; hence, the corresponding series converges in the left halfplane.

2.5. **Classes of real equivalence.** Our goal is to express the Dulac map of a real saddle-node through F_{norm} and the formal normal form of the saddle-node. There is no chance to do this explicitly, because there is no preferred chart on the image cross-section. For a real saddle-node this chart is defined up to a left composition with a real holomorphic germ.

This motivates the following definition.

Definition 3. *Two germs f and g on the positive real axis holomorphic at 0 are (left) real equivalent if*

$$f = h \circ g, \ \ h : (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0). \tag{7}$$

We stress that the germs are considered at zero on the positive real axis. Our goal now is to find the class of real equivalence of the germ g^u.

Let S be the operator of complex conjugacy:

$$(Sf)(z) = \overline{f(\bar{z})} \tag{8}$$

Proposition 1. *Two germs f and g are (left) real equivalent iff*

$$I := f^{-1} \circ Sf = g^{-1} \circ Sg. \tag{9}$$

Proof. If $f = h \circ g$, then $Sf = h \circ Sg$, and

$$f^{-1} \circ Sf = g^{-1} \circ h^{-1} \circ h \circ Sg = g^{-1} \circ Sg,$$

If (9) holds, then

$$h = g \circ f^{-1} = Sg \circ (Sf)^{-1} = S(g \circ f^{-1}) = Sh.$$

Hence, h is real. □

Definition 4. *The germ $f^{-1} \circ Sf$ is called the invariant of the class of the real equivalence of f.*

2.6. **Improved versions of Theorem 1.** The Dulac map of any complex saddle-node has the form [I]:

$$\Delta = g^u \circ \Delta_{st} \circ F^u = g^l \circ \Delta_{st} \circ F^l \tag{10}$$

where g^u and g^l are holomorphic germs, F^u and F^l are defined above, and Δ_{st} is the Dulac map for the system

$$\dot{z} = \frac{z^{k+1}}{1 + az^k}, \ \dot{w} = -w. \tag{11}$$

The system (11) is the formal orbital normal form for v. Changing the scale on $\Gamma^- : w \to cw$, one may achieve that

$$\Delta_{st} = \exp \circ t_{k,a}, \tag{12}$$

see Figure 2. The sectorial normalization H^u conjugates the original vector field with a standard one in some domain that contains a germ of an open real right halfplane at zero. The map H^u brings a horizontal sector $w = \text{const}$ that contains Γ^+ to a curvilinear sector. The map π along the orbits of the normalized vector field brings the latter sector to a horizontal one. The map $\pi \circ H^u|_{\Gamma^+}$ is normalizing for the monodromy transformation; it is a component F^u of the normalizing cochain. The map H^u brings a vertical disc $z = \text{const}$ that contains Γ^- to another vertical disc. The map $g = g^u$ equals to $(H^u)^{-1}$ restricted to the second disc.

For a real saddle-node ,

$$F^l = SF^u, \; g^l = Sg^u.$$

This easily follows from Appendix II to Theorem 2 in [I].

Theorem 2. *Let v be a real saddle-node formally orbitally equivalent to the germ* (11). *Let*

$$F^u \circ (SF^u)^{-1} = t_{k,a}^{-1} \circ \delta_0 \circ t_{k,a},$$

where $SF^u = F^l$ is from (5)*, δ_0 has the form* (6)*, and $t_{k,a}$ is from* (3)*. Then the invariant of the class of $\mathbb{R}$-equivalence of the germ g^u is equal to*

$$(g^u)^{-1} \circ (Sg^u) = \exp \circ \delta_0 \circ \ln .$$

We call the germ

$$\tilde{\delta}_0 = \exp \circ \delta_0 \circ \ln$$

the renormalized Martinet-Ramis modulus.

It is a parabolic germ $(\mathbb{C}, 0) \to \mathbb{C}, 0) : \tilde{\delta}_0'(0) = 1$, as proved below.

Theorem 3. *Let the renormalized Martinet-Ramis modulus of a saddle-node germ v admit a compositional square root extraction:*

$$\tilde{\delta}_0 = j \circ j, \; j : (\mathbb{C}, 0) \to \mathbb{C}, 0), j'(0) = 1.$$

Then the factor g^u in (10) *is real equivalent to Sj. In other words, in an appropriate real chart on the image cross-section, the Dulac map of v equals*

$$\Delta = Sj \circ \Delta_{st} \circ F^u.$$

Definition 5. *A saddle-node v is absolutely real if $F^u \equiv F^l$ on* $(\mathbb{R}^+, 0)$.

Corrolary 1. *Let v be a germ of an absolutely real saddle-node . Then in an appropriate real chart on the image cross-section the Dulac map for v has the form*

$$\Delta = \Delta_{st} \circ F^u_{norm} = \Delta_{st} \circ F^l_{norm}. \tag{13}$$

Proof. For the absolutely real saddle-node $\delta_0 = \text{id}$. Hence, $(g^u)^{-1} \circ (Sg^u) = \text{id}$, and the identity belongs to the left $\mathbb{R}$-equivalence class of g^u. □

We can now find a formal representative of the $\mathbb{R}$-equivalence class of g^u.

Any parabolic germ $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ admits a formal square root extraction:

$$[f] = [j] \circ [j],$$

where $[f]$ is a convergent Taylor series for the germ f, and $[j]$ is a formal Taylor series with the linear part identity, divergent in general. Indeed, a formal normal form of a parabolic germ is a phase flow transformation, and such a transformation admits a compositional root extraction of any degree. On the contrary, the set of parabolic germs that admit a holomorphic square root extraction has in a sense codimension infinity. Indeed, the Ecalle-Voronin modulus for such germs have to commute not only with a shift by $2\pi i$, but rather by πi, see [V] for details.

Theorem 4. *Let $[j]$ be a formal Taylor series, compositional square root of $[\tilde{\delta}_0]$, divergent in general:*

$$[\tilde{\delta}_0] = [j] \circ [j].$$

Then there exists a real formal series $[h]$, also divergent in general, such that in formula (10)

$$[g^u] = [h] \circ [Sj].$$

3. Proofs of the main theorems

3.1. **Proof of Theorem 2.** By (10) we have

$$g^u \circ \Delta_{st} \circ F^u = g^l \circ \Delta_{st} \circ F^l.$$

But for real saddle-nodes,

$$g^l = Sg^u, \ F^l = SF^u.$$

Hence,

$$(g^u)^{-1} \circ (Sg^u) = \Delta_{st} \circ F^u \circ (SF^u)^{-1} \circ (\Delta_{st})^{-1}.$$

But

$$\Delta_{st} = \exp \circ t_{k,a}, \ F^u \circ (SF^u)^{-1} = (t_{k,a})^{-1} \circ \delta_0 \circ t_{k,a}.$$

Hence

$$(g^u)^{-1} \circ (Sg^u) = \exp \circ \delta_0 \circ \ln .$$

Theorem 2 is proved.

Remark 1. *The map $\delta_0 - \zeta$ is expressed as an exponential series* (6) *with a zero free term. Hence,*

$$\exp \circ \delta_0 \circ \ln = \zeta \exp \sum_1^\infty a_n \zeta^n,$$

the series in the r. h. s. converges at zero. This germ is holomorphic at zero and parabolic, as claimed after the statement of Theorem 2.

3.2. **Proof of Theorem 3.** The invariants of classes of real equivalence are anti-real in the following sense.

Definition 6. *A germ* $I : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *is* anti-real *if* $SI = I^{-1}$.

The reason for the name is that the conjugate of the real germ is the same germ, and for the anti-real germ it is the inverse germ.

Example 1. *The monodromy map of a real saddle-node is anti-real. Indeed, the complex conjugacy map of the plane brings the monodromy map of the saddle-node to a complex conjugate one. At the same time, this is a monodromy map that corresponds to the same loop, but oppositely oriented. Hence, the conjugate to the original monodromy map is inverse to it, see* [I] *for details.*

Note that the invariants of the classes of $\mathbb{R}$-equivalence are anti-real. Indeed, if $I = f^{-1} \circ Sf$, then

$$SI = (Sf)^{-1} \circ f = I^{-1}.$$

Here we used the obvious relation: $S(f^{-1}) = (Sf)^{-1}$.

Lemma 1. *Suppose that an anti-real parabolic germ admits a compositional square root extraction. Then it is the invariant of the class of real equivalence of its conjugated square root.*

Proof. Let $I = j \circ j$. Then I is the invariant of the class of real equivalence of the germ Sj. Indeed, by assumption, $SI = I^{-1}$. Hence, $Sj \circ Sj = j^{-1} \circ j^{-1}$. As the square root extraction is unique, we conclude that j is anti-real as well. Hence, $(Sj)^{-1} \circ j = j \circ j = I$. □

By assusmption of Theorem 3, $\tilde{\delta}_0$ satisfies the assumption of the lemma. Let $j \circ j = \tilde{\delta}_0$. Then Sj belongs to the class of the $\mathbb{R}$-equivalence of g^u. This proves Theorem 3.

The proof of Theorem 4 follows the same lines as that of Theorem 3, only the holomorphic germs are replaced by formal series.

4. An inverse problem for classes of $\mathbb{R}$-equivalence

A natural question arises: is any anti-real germ an invariant of some class of left real equivalence?

Theorem 5. *Any anti-real parabolic germ is an invariant of some class of left real equivalence of some holomorphic germ with a positive derivative.*

I could not find a proof of this statement in frame of the theory of one-dimensional germs. The theorem is proved below with the use of all the above constructions. Namely, given an anti-real germ we construct a saddle-node whose Martinet-Ramis modulus is related to the given anti-real germ as above. Then the germ g^u will be the desired one.

Proof. Consider the class of the saddle-nodes with the orbital formal normal form

$$\dot{z} = z^2,\ \dot{w} = -w. \tag{14}$$

The rectifying map for the vector field z^2 is $t_{1,0} = -\frac{1}{z}$. Now consider the given anti-real parabolic germ $I = \zeta + \sum_2^\infty a_n \zeta^n$. Take

$$\delta_0 = \ln \circ I \circ \exp = \ln(\exp \zeta + \sum_2^\infty a_n \exp n\zeta) =$$

$$\zeta + \ln(1 + \sum_2^\infty a_n \exp(n-1)\zeta) = \zeta + \sum_1^\infty b_n \exp n\zeta.$$

By the realization theorem for the Martinet-Ramis moduli, there exists a real germ v formally orbitally equivalent to (14) for which the normalizing cochain of the monodromy map has the properties:

$$F^u = SF^l,\ F^u \circ (F^l)^{-1} = t_{1,0}^{-1} \circ \delta_0 \circ t_{1,0}.$$

For this saddle-node

$$\Delta = g^u \circ \Delta_{st} \circ F^u,\ \Delta_{st} = \exp\left(-\frac{1}{z}\right),$$

and

$$(g^u)^{-1} \circ Sg^u = \exp \circ \delta_0 \circ \ln = I.$$

The derivative $(g^u)'(0)$ is positive by the Appendix II to Theorem 2 in [I]. □

It is interesting to note that the class of real equivalence thus constructed does not depend on the choice of the formal normal form (11).

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